

The University of Chicago

GENERALIZED INTERSECTOR VARIETIES

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1939

By

VIRGIL NELSON ROBINSON

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GENERALIZED INTERSECTOR VARIETIES

1. Introduction. The definition of intersector curves was introduced for curves on ruled surfaces¹ in three-dimensional space by Lane, who later extended the definition and developed the theory to include curves on general varieties in a linear space of n -dimensions.² The theory of intersector curves was generalized by Cook³ who defined intersector surfaces on congruences in three-space. A very general theory has since been presented by Borgman⁴ in which he defined intersector varieties on varieties in n -dimensional space. In his definition, Borgman considered two complementary linear spaces, each of which generates a variety. A point in one of these spaces also generates a variety. It is the tangent space at a point of this latter variety which is fundamental in Borgman's definition. This space, subject to certain conditions, he termed the intersector tangent space.

The theory developed in the present paper is, in a sense, a generalization of that of Borgman. Instead of a single point in one of the complementary spaces, as in Borgman's theory, an arbitrary linear subspace is here considered as generating a va-

¹E. P. Lane, Ruled surfaces with generators in one-to-one correspondence, Transactions of the American Mathematical Society, vol. 25 (1923), p. 281.

²E. P. Lane, Projective Differential Geometry of Curves and Surfaces, The University of Chicago Press, 1932, p. 278.

³A. J. Cook, Pairs of rectilinear congruences with generators in one-to-one correspondence, Transactions of the American Mathematical Society, vol. 32 (1930), p. 31.

⁴W. M. Borgman, Intersector varieties in hyperspace, Doctor's dissertation, University of Chicago, 1934.

riety. Borgman's intersector tangent space at a point of a variety is here replaced by an intersector tangent space along a linear subspace of a variety.

In Section 2, a completely integrable system of linear partial differential equations is set up as the basis of the projective theory. This is a general system in the sense that the number of independent variables and the number of dependent variables are arbitrary.

Generalized intersector varieties are defined geometrically in Section 3 and the equations of their generators are written for convenience in both homogeneous and non-homogeneous coordinates. It is also noted in Section 3, that by suitable specialization of the present theory, the theories of the intersector curves, surfaces, and varieties as developed by Lane, Cook, and Borgman result.

In the definition of generalized intersector varieties, certain mild restrictions are assumed on the dimensionalities of the spaces concerned. The reasons for assuming these restrictions are treated in Section 4.

Section 5 is devoted to the analytical characterization of generalized intersector varieties. In this section, necessary and sufficient conditions that a variety be an intersector variety are developed. These conditions are in the form of a system of first-order partial differential equations. The existence of solutions of these equations is discussed in Section 6. To simplify the discussion, the intersector varieties are divided into four classes. For the first of these classes, the present theory becomes specialized and the intersector varieties reduce to a type discussed by Borgman. In the second class, intersector varieties exist, provided that certain integrability conditions are satisfied, and depend upon a number of arbitrary constants which is

determined. Solutions in the third and fourth cases always exist and depend upon a determined number of arbitrary functions. The section closes with a table summarizing the results.

In the first six sections, only intersector varieties in the first of two complementary spaces have been considered. By interchanging the two complementary spaces, the intersector varieties in the second complementary space are discussed, and the theory developed as before. Under certain conditions, the intersector varieties in the two complementary spaces are defined to be associated intersector varieties, and under other conditions to be symmetrical associated intersector varieties. Section 7 is devoted to this discussion and closes with necessary and sufficient conditions that families of intersector varieties be symmetrical associated families.

In Section 8, a local coordinate system is defined, by means of which the form of the equations of the loci of the tangent intersector spaces is derived.

2. The differential equations. Let us consider a linear projective space of n dimensions S_n determined by $n+1$ linearly independent points $x_1, x_2, \dots, x_{n+1}$. The $n+1$ homogeneous coordinates of a single point in S_n will be designated by superscripts. Thus the point x_1 has the coordinates $x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(n+1)}$. Let us assume that the coordinates of the $n+1$ points in S_n are all single-valued analytic functions of the same h independent variables $u_1, u_2, \dots, u_h$, where $0 < h \leq n$. The coordinates of these $n+1$ independent points can be arranged in the form of a square matrix of rank $n+1$. Hence it is possible¹ to determine the coefficients a_{j1q} of a system of equations of the form

¹W. M. Borgman, loc. cit., p. 4.

$$(2.1) \quad \frac{\partial x_j}{\partial u_q} = \sum_{i=1}^{n+1} a_{jiq} x_i \quad (j = 1, 2, \dots, n+1; q = 1, 2, \dots, h),$$

so that $x_1^{(1)}, x_2^{(1)}, \dots, x_{n+1}^{(1)}$ ($i = 1, 2, \dots, n+1$) will be $n+1$ independent sets of solutions which satisfy the integrability conditions of the system (2.1), namely,

$$(2.2) \quad \frac{\partial a_{jiq}}{\partial u_p} + \sum_{t=1}^{n+1} a_{jtp} a_{tiq} = \frac{\partial a_{jip}}{\partial u_q} + \sum_{t=1}^{n+1} a_{jtp} a_{tiq} \\ (i, j = 1, 2, \dots, n+1; p, q = 1, 2, \dots, h).$$

Conversely, if the system of equations (2.1) is given, along with the integrability conditions (2.2), then the system (2.1) has $n+1$ linearly independent sets of solutions.¹ Therefore, these sets of particular solutions can be used as the $n+1$ homogeneous projective coordinates of the $n+1$ linearly independent points, and any other set is a projective transform of this set.

3. Definition of generalized intersector varieties. Let the $n+1$ linearly independent points $x_1, x_2, \dots, x_{n+1}$, determining the ambient space S_n , be divided into two sets

$$x_1, x_2, \dots, x_m; \quad x_{m+1}, x_{m+2}, \dots, x_{n+1}.$$

These sets determine complementary linear spaces S_{m-1} and S_{n-m} respectively. Since each point has coordinates depending upon h independent variables $u_1, u_2, \dots, u_h$, each point has h degrees of freedom and generates a variety V_h of h dimensions. Similarly, the spaces S_{m-1} and S_{n-m} generate varieties V_{m+h-1} and V_{n-m+h} of dimensionalities $m+h-1$ and $n-m+h$ respectively.

In the complementary space S_{m-1} , any space S_f of f dimensions is determined by $f+1$ points $y_0, y_1, \dots, y_f$ given by

¹G. M. Green, The linear dependence of functions of several variables, and completely integrable systems of homogeneous linear partial differential equations, Transactions of the American Mathematical Society, vol. 17 (1916), p. 514.

$$(3.1) \quad y_p = \sum_{j=1}^m \lambda_{pj} x_j \quad (p = 0, 1, \dots, f),$$

where λ_{pj} are analytic functions of the independent variables $u_1, u_2, \dots, u_h$. With the simplex of reference chosen as in Section 2, and by a suitable choice of unit point, the coordinates of a point y_p , given by equations (3.1), may be taken as

$$(3.2) \quad \lambda_{p1}, \lambda_{ps}, \dots, \lambda_{pm} \quad (p = 0, 1, \dots, f).$$

Since these are homogeneous coordinates, not all zero, by assuming $\lambda_{p1} \neq 0$ and letting

$$(3.3) \quad u_{pj} = \frac{\lambda_{pj}}{\lambda_{p1}} \quad (p = 0, 1, \dots, f; j = 1, 2, \dots, m),$$

the coordinates (3.2) may be written in non-homogeneous form

$$(3.4) \quad 1, u_{ps}, u_{ps}, \dots, u_{pm} \quad (p = 0, 1, \dots, f).$$

It will sometimes be useful to use the coordinates in the latter form.

The space S_f determined by the points y_p given by equations (3.1) generates a variety V_{f+h} which necessarily lies on the variety V_{m+h-1} . There is an $[(f+1)(h+1)-1]$ -dimensional linear space $S_{(f+1)(h+1)-1}$ which is tangent to the variety V_{m+h-1} along the generating linear space¹ S_f . Let us suppose that the tangent space $S_{(f+1)(h+1)-1}$ intersects the complementary space S_{n-m} in a linear space $S_{(f+1)(h+1)-(m+1)+k}$ of $(f+1)(h+1)-(m+1)+k$ dimensions, where k is an integer. Then the locus V_{h+f} of the space S_f under these conditions may be defined to be an intersector variety of index k on the variety V_{m+h-1} with respect to the variety V_{n-m+h} .

It is easily seen that the intersector varieties so de-

¹E. P. Lane, Projective Differential Geometry of Curves and Surfaces, p. 273.

defined have as special cases the intersector curves of Lane, the intersector surfaces of Cook, and the intersector varieties of Borgman. For, this definition of intersector varieties reduces to that given by Borgman¹ in case $f=0$, to that considered by Cook² in case

$$f = 0, \quad n = 3, \quad h = 2, \quad m = 2, \quad k = 1,$$

to the intersector curves of Lane³ for

$$f = 0, \quad h = 1, \quad k = m-1,$$

and to Lane's original intersector curves on a ruled surface⁴ in case

$$f = 0, \quad n = 3, \quad h = 1, \quad k = 1, \quad m = 2.$$

4. Restrictions on the constants. Certain restrictions are imposed upon the constants n, m, h, k, f . The nature of these restrictions and the reasons for imposing them will now be explained.

Since the space S_f lies within the space S_{m-1} , we have

$$(4.1) \quad f \leq m-1.$$

If $k = 0$, every variety generated by the space S_f would be an intersector variety since a space $S_{(f+1)(h+1)-1}$ always intersects a space S_{n-m} in a space $S_{(h+1)(f+1)-(m+1)}$. Hence, to avoid a trivial situation, we assume

$$(4.2) \quad k > 0.$$

If $h = 0$, the tangent space $S_{(f+1)(h+1)-1}$ coincides with

¹Loc. cit., p. 8.

²Loc. cit., p. 31.

³Loc. cit., p. 278.

⁴Loc. cit., p. 281.

the space S_f . Thus the space S_f would not generate a variety at all but would remain fixed. Therefore we assume

$$(4.3) \quad h > 0.$$

If $k = m$, the tangent space $S_{(f+1)(h+1)-1}$ is the space $S_{(f+1)(h+1)-(m+1)+k}$ which is the intersection space lying entirely in S_{n-m} . Thus the space S_f would lie in the space S_{n-m} but was chosen to be in the space S_{m-1} . Since the points are all linearly independent, to avoid a contradiction we take

$$(4.4) \quad k < m.$$

This inequality applies for, if $k > m$, by similar reasoning, the space S_f would be contained in the space S_{n-m} which is a contradiction.

Since the variety V_{m+h-1} is generated by the space S_{m-1} , it must lie in the ambient space S_n . Hence

$$(4.5) \quad m + h - 1 \leq n.$$

Similarly, the variety V_{n-m+h} lies in the space S_n and therefore we have

$$(4.6) \quad n - m + h \leq n.$$

The intersection space $S_{(h+1)(f+1)-(m+1)+k}$ must lie in both spaces S_{n-m} and $S_{(f+1)(h+1)-1}$. It follows that

$$(4.7) \quad (h+1)(f+1) - (m+1) + k \leq n - m,$$

$$(4.8) \quad (h+1)(f+1) - (m+1) + k \leq (h+1)(f+1) - 1.$$

In fact, by inequality (4.4), the inequality (4.8) may be rewritten using the "strict inequality" sign.

The value of k must be large enough so that the spaces S_{n-m} and $S_{(f+1)(h+1)-1}$ actually do intersect. Therefore

$$(4.9) \quad (f+1)(h+1) - (m+1) + k \geq 0.$$

By adding the two inequalities (4.5) and (4.6) and adding $1 - n$ to both members of the result, we have,

$$(4.10) \quad 2h \leq n + 1.$$

Combining inequalities (4.4) and (4.9), we find

$$(4.11) \quad 1 \leq m - k \leq (h+1)(f+1) - 1.$$

The inequality (4.11) will be used in Section 6 to classify the intersector varieties.

Combining inequalities (4.2) and (4.4) we have

$$(4.12) \quad m > 1.$$

Inequalities (4.2), (4.3), and (4.7) imply

$$(4.13) \quad 2f + 2 \leq n,$$

and hence

$$(4.14) \quad n > 1.$$

5. Analytic intersector varieties. Equations (3.1) are the parametric equations of a space S_f in the complementary space S_{m-1} . A necessary and sufficient condition that the space S_f generate an intersector variety according to the definition in Section 3 will now be determined.

For this purpose, let us observe that the linear tangent space along the space S_f in the space S_{m-1} is a space of $(f+1)(h+1) - 1$ dimensions and is determined by the $(f+1)(h+1)$ points¹

¹E. P. Lane, Projective Differential Geometry of Curves and Surfaces, p. 273.

$$(5.1) \quad y_p, \frac{\partial y_p}{\partial u_q} \quad (p = 0, 1, \dots, f; q = 1, 2, \dots, h),$$

where the points y_p are given by equations (3.1). By the definition of intersector varieties, a certain number, namely

$$(h + 1)(f + 1) - (m + 1) + (k + 1),$$

of linearly independent points among the points (5.1) also lie in the second complementary space S_{n-m} . Let these points be designated as linear combinations of the points (5.1) by

$$(5.2) \quad \phi_s = \sum_{p=0}^f \alpha_{sp0} y_p + \sum_{p=0}^f \sum_{q=1}^h \alpha_{spq} \frac{\partial y_p}{\partial u_q} \quad (s = 0, 1, \dots, N),$$

where

$$N = (h + 1)(f + 1) - (m + 1) + k,$$

and where the coefficients α are analytic functions of the independent variables $u_1, u_s, \dots, u_h$. The matrix of the coefficients α in equation (5.2) is

$$(5.3) \quad (\alpha_{s00} \alpha_{s10} \dots \alpha_{sfo} \alpha_{s01} \alpha_{s11} \dots \alpha_{sfi} \alpha_{s0s} \alpha_{s1s} \dots \alpha_{sfs} \dots \alpha_{sfh}) \\ (s = 0, 1, \dots, N).$$

Each row of the matrix (5.3) has $h + 1$ groups, each of $f + 1$ elements. Consequently the matrix (5.3) has $(h + 1)(f + 1) - (m - k)$ rows and $(f + 1)(h + 1)$ columns. Since the points ϕ_s are all linearly independent, the matrix (5.3) must be of rank $(h + 1)(f + 1) - (m - k)$.

Since the points ϕ_s all lie in the space S_{n-m} , the points ϕ_s may be written as

$$(5.4) \quad \phi_s = [x_{m+1}, x_{m+s}, \dots, x_{n+1}],$$

where the brackets represent a linear combination of the points $x_{m+1}, \dots, x_{n+1}$.

After substitution of the values given for the points y_p by equations (3.1) into the equations (5.2) the latter become

$$(5.5) \quad \phi_s = \sum_{p=0}^f \sum_{j=1}^m \left[\alpha_{spo} \lambda_{pj} x_j + \sum_{q=1}^h \alpha_{spq} \left(\frac{\partial \lambda_{pj}}{\partial u_q} x_j + \lambda_{pj} \frac{\partial x_j}{\partial u_q} \right) \right]$$

$$(s = 0, 1, \dots, N).$$

After substituting into equations (5.5) the values of $\partial x_j / \partial u_q$ as given by equations (2.1) we have

$$(5.6) \quad \phi_s = \sum_{p=0}^f \sum_{j=1}^m \left[\alpha_{spo} \lambda_{pj} x_j + \sum_{q=1}^h \alpha_{spq} \left(\frac{\partial \lambda_{pj}}{\partial u_q} x_j + \sum_{i=1}^{n+1} \lambda_{pj} a_{jiq} x_i \right) \right]$$

$$(s = 0, 1, \dots, N).$$

Since, by equations (5.4), the right member of equations (5.6) cannot involve the points $x_1, x_2, \dots, x_m$, the coefficients of $x_1, x_2, \dots, x_m$ are zero. Then for a particular value of j we have

$$(5.7) \quad \sum_{p=0}^f \left[\alpha_{spo} \lambda_{pj} + \sum_{q=1}^h \alpha_{spq} \left(\frac{\partial \lambda_{pj}}{\partial u_q} + \sum_{t=1}^m \lambda_{pt} a_{jtq} \right) \right] = 0$$

$$(s = 0, 1, \dots, N; j = 1, 2, \dots, m).$$

Let us now make the following abbreviations:

$$(5.8) \quad \begin{aligned} L_{pj}^{(0)} &= \lambda_{pj}, \\ L_{pj}^{(q)} &= \frac{\partial \lambda_{pj}}{\partial u_q} + \sum_{t=1}^m a_{jtq} \lambda_{pt} \end{aligned}$$

$$(j = 1, 2, \dots, m; p = 0, 1, \dots, f; q = 1, 2, \dots, h).$$

With the aid of these abbreviations, equations (5.7) reduce to

$$(5.9) \quad \sum_{p=0}^f \left[\alpha_{spo} L_{pj}^{(0)} + \sum_{q=1}^h \alpha_{spq} L_{pj}^{(q)} \right] = 0$$

$$(s = 0, 1, \dots, N; j = 1, 2, \dots, m),$$

for a particular value of j . The matrix of equations (5.9) is ex-

actly the matrix (5.3) which has rank $N + 1$, that is,
 $(h + 1)(f + 1) - (m - k)$. Let us suppose, for convenience of notation, that the square matrix of the first $N + 1$ columns is different from zero. Then the first $(h + 1)(f + 1) - (m - k)$ of the variables, namely

$$(5.10) \quad L_{0j}^{(0)}, L_{1j}^{(0)}, \dots, L_{hj}^{(0)}, L_{0j}^{(1)}, \dots, L_{hj}^{(1)}, \dots, L_{0j}^{(A)}, \dots, L_{Bj}^{(A)},$$

can be found uniquely as homogeneous linear functions of the remaining $(m - k)$ variables¹ for any particular value of j . Let the variables thus determined be those shown in (5.10), where

$$(5.11) \quad (A + 1)(f + 1) - (f - B) = (h + 1)(f + 1) - (m - k).$$

Hence in the double array

$$(5.12) \quad \begin{array}{ccccccc} L_{0j}^{(0)} & L_{0j}^{(1)} & \dots & L_{0j}^{(A)} & \dots & L_{0j}^{(h)} & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & L_{Bj}^{(A)} & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ L_{fj}^{(0)} & L_{fj}^{(1)} & \dots & L_{fj}^{(A)} & \dots & L_{fj}^{(h)} & \end{array}$$

the last element which can be thus determined is given by the coordinates $(A + 1, B + 1)$, where $A + 1$ designates the column and $B + 1$ the row in which the element is found. All elements in columns preceding the $(A + 1)^{\text{th}}$ column and all elements in the $(A + 1)^{\text{th}}$ column to and including the element $L_{Bj}^{(A)}$ can be found as linear combinations of the remaining elements.

We may write the solutions for the expressions (5.10) in the form

¹L. E. Dickson, Modern Algebraic Theories, Benjamin H. Sanborn and Co., 1926, p. 61.

$$(5.15) \quad \begin{vmatrix} L_{B+1,b}^{(A)} & \dots & L_{f,b}^{(h)} & L_{Eb}^{(D)} \\ L_{B+1,b+1}^{(A)} & \dots & L_{f,b+1}^{(h)} & L_{E,b+1}^{(D)} \\ \cdot & \cdot & \cdot & \cdot \\ L_{B+1,b+m-k}^{(A)} & \dots & L_{f,b+m-k}^{(h)} & L_{E,b+m-k}^{(D)} \end{vmatrix} = 0,$$

$$(b = 1, 2, \dots, k; D = 0, 1, \dots, A; E = 0, \dots, f).$$

Equations (5.15) follow from the fact that the space S_f , determined by equations (3.1), generates an intersector variety.

Conversely, let us assume that the functions λ_{pj} , which in equations (3.1) define the locus of the space S_f , are solutions of equations (5.15), the functions $L_{pj}^{(q)}$ being defined by equations (5.8). With this hypothesis, it will be proved that the space S_f generates an intersector variety.

Equations (5.15) imply that the functions β can be found such that equations (5.13) are satisfied. But equations (5.13) are of the form of equations (5.9) with a matrix of coefficients of rank $(h+1)(f+1) - (m-k)$, for there is one $(h+1)(f+1) - (m-k)$ -rowed determinant in this matrix the value of which is unity. The elements of the matrix (5.3) can then be used to define $(h+1)(f+1) - (m-k)$ points ϕ_s given by (5.2). These points will be linearly independent, will lie in the tangent space $S_{(h+1)(f+1)-1}$, and will also lie in the space S_{n-m} . Therefore the two spaces $S_{(h+1)(f+1)-1}$ and S_{n-m} have a linear space $S_{(h+1)(f+1)-(m-k)-1}$ in common, and hence the variety V_{h+f} generated by the space S_f is an intersector variety. Thus the following theorem has been proved:

Theorem 1. The space S_f , as defined by equations (3.1), generates an intersector variety if and only if the functions λ_{pj}

satisfy the differential equations (5.15) where the L's are de-
defined by equations (5.8).

In the theory above, the homogeneous coordinates (3.2) have been used to define the space S_f given by equations (3.1). However, by the transformation (3.3), these coordinates may be changed to non-homogeneous coordinates of the form (3.4), in this case equations (3.1) become

$$(5.16) \quad y_p = x_1 + \sum_{j=s}^m U_{pj} x_j, \quad (p = 0, 1, \dots, f).$$

By the aid of equations (5.16), the argument may be carried through as before, where, instead of the abbreviations (5.8), we now have

$$(5.17) \quad \begin{aligned} L_{p1}^{(0)} &= 1, \\ L_{pd}^{(0)} &= U_{pd}, \\ L_{p1}^{(q)} &= a_{11q} + \sum_{t=s}^m a_{1tq} U_{pt}, \\ L_{pd}^{(q)} &= a_{d1q} + \frac{\partial U_{pd}}{\partial u_q} + \sum_{t=s}^m a_{dtq} U_{pt} \end{aligned}$$

$$(d = 2, 3, \dots, m; p = 0, 1, \dots, f; q = 1, 2, \dots, h).$$

Equations (5.9), ..., (5.15) remain unchanged and theorem 1 follows as before. Hence, as the differential equations of the intersector variety, we may take equations (5.15), with the values of the L's defined either as in equations (5.8) or as in equations (5.17).

6. Existence of intersector varieties. For convenience, we shall divide the discussion of existing intersector varieties into four classes, according to the different possibilities in inequality (4.11),

- (a) $l = m - k = hf + h + f.$
 (b) $l = m - k < hf + h + f.$
 (c) $l < m - k = hf + h + f.$
 (d) $l < m - k < hf + h + f.$

These four possibilities which we have labeled cases (a), (b), (c), and (d), respectively, will now be discussed in turn.

Case (a). $l = m - k = hf + h + f.$ By inequality (4.3), we have $h > 0$, and since f is necessarily an integer or zero, it must be zero in order to satisfy the condition for case (a). Hence the space S_f reduces to a single point. This is the case discussed in detail by Borgman¹ who showed that the intersector varieties always exist and depend upon $m - 1$ arbitrary constants. Therefore, under case (a), there is an $(m - 1)$ - parameter family of intersector varieties.

Case (b). $l = m - k < hf + h + f.$ In this case, the equations (5.15) become

$$(6.1) \quad \begin{vmatrix} \binom{h}{L_{f1}} & \binom{D}{L_{E1}} \\ \binom{h}{L_{fd}} & \binom{D}{L_{Ed}} \end{vmatrix} = 0 \quad \begin{array}{l} (d = 2, \dots, m; D = 0, \dots, h; \\ E = 0, \dots, f) \\ (D, E) \neq (h, f), \end{array}$$

where the variable subscripts and superscripts take the values indicated, except that the conditions $D = h$, $E = f$ cannot hold simultaneously, that is, for $D = h$ we have $E \leq h - 1$, and for $E = f$, we have $D \leq h - 1$. This follows from the fact that inspection of (5.10), shows that in this case all the variables except one may be found in terms of the one remaining variable. The determinants (6.1) are equivalent to the determinants

¹Loc. cit., p. 19.

$$(6.2) \quad \begin{vmatrix} L_{o1}^{(o)} & L_{E1}^{(D)} \\ L_{od}^{(o)} & L_{Ed}^{(D)} \end{vmatrix} = 0 \quad \begin{matrix} (d=2, \dots, m; D=1, \dots, h; E=0, \dots, f) \\ (D, E) \neq (h, f). \end{matrix}$$

Upon substitution of the values of L given in equations (5.17), the determinants (6.2) become

$$(6.3) \quad \begin{vmatrix} 1 & \sum_{t=1}^m a_{1t} D U_{Et} \\ U_{od} & \frac{\partial U_{Ed}}{\partial u_D} + \sum_{t=1}^m a_{dt} D U_{Et} \end{vmatrix} = 0$$

$$(d=2, 3, \dots, m; D=1, 2, \dots, h; E=0, 1, \dots, f; (D, E) \neq (h, f); U_{p1}=1).$$

Expanding the determinants (6.3), we have

$$(6.4) \quad \frac{\partial U_{Ed}}{\partial u_D} = -a_{d1D} + a_{11D} U_{od} + \sum_{r=2}^m (a_{drD} - a_{1rD} U_{od}) U_{Er} = 0$$

$$(d=2, 3, \dots, m; D=1, 2, \dots, h; E=0, 1, \dots, f; (D, E) \neq (h, f)).$$

In order that equations (6.4) have analytic solutions,¹ it is necessary and sufficient that they be completely integrable, that is, that the following integrability conditions be satisfied:

$$(6.5) \quad \frac{\partial^2 U_{Ed}}{\partial u_D \partial u_q} = \frac{\partial^2 U_{Ed}}{\partial u_q \partial u_D}$$

$$(d=2, 3, \dots, m; D, q=1, 2, \dots, h; D < q; E=0, 1, \dots, f).$$

Upon differentiating equations (6.4) to compute the derivatives appearing in equations (6.5), and making use of equations (2.2) as well as equations (6.4), the integrability conditions (6.5) can be reduced to the form

¹Edouard Goursat, Leçons sur l'Intégration des Equations aux Derivées Partielles du Premier Ordre, Librairie Scientifique, J. Hermann, 1921, p. 104.

$$\begin{aligned}
 (6.6) \quad & \sum_{t=1}^m \sum_{i=m+1}^{n+1} (a_{1iq} a_{1td} - a_{1id} a_{1tq}) U_{od} U_{Et} \\
 & + \sum_{t=1}^m \sum_{i=m+1}^{n+1} (a_{d1D} a_{1tq} - a_{diq} a_{1tD}) U_{Et} = 0 \\
 & (d=2, 3, \dots, m; D, q=1, 2, \dots, h; D < q; E=0, 1, \dots, f).
 \end{aligned}$$

In view of the fact that $U_{E1} = 1$, equations (6.6) may be written in the form

$$\begin{aligned}
 (6.7) \quad & \sum_{t=s}^m \sum_{i=m+1}^{n+1} (a_{1iq} a_{1td} - a_{1id} a_{1tq}) U_{od} U_{Et} \\
 & + \sum_{t=s}^m \sum_{i=m+1}^{n+1} (a_{d1D} a_{1tq} - a_{diq} a_{1tD}) U_{Et} \\
 & + \sum_{i=m+1}^{n+1} (a_{1iq} a_{11D} - a_{1id} a_{11q}) U_{od} \\
 & + \sum_{i=m+1}^{n+1} (a_{d1D} a_{11q} - a_{diq} a_{11D}) = 0 \\
 & (d=2, 3, \dots, m; D, q=1, 2, \dots, h; D < q; E=0, 1, \dots, f).
 \end{aligned}$$

If equations (6.7) are satisfied identically or conditionally, the system of equations (6.4) has solutions. For, if equations (6.7) are satisfied identically, the existence theorem¹ for partial differential equations of the type to which equations (6.7) belong, states that there exists one and only one solution

$$U_{O2}(u_1, u_2, \dots, u_h), U_{O3}, \dots, U_{fm}(u_1, u_2, \dots, u_h)$$

of the system (6.7) through an initial point

$$(u_1^0, u_2^0, \dots, u_h^0; U_{O2}^0, \dots, U_{fm}^0).$$

The first h coordinates of this point determine a particular gener-

¹Ibid., p. 105.

ator S_{m-1} on the variety V_{m+h-1} , and the remaining $(f+1)(m-1)$ coordinates determine the position of the space S_f in that generator. The latter set of coordinates is given in non-homogeneous form as exhibited in equations (3.4). Since there is a $(f+1)(m-f-1)$ parameter family of spaces¹ S_f in the generating space S_{m-1} , we have proved

Theorem 2. In case (b), if equations (6.7) are satisfied identically, then there exists an $(f+1)(m-f-1)$ -parameter family of analytic intersector varieties V_{f+h} on the variety V_{m+h-1} with respect to the variety V_{n-m+h} , and conversely.

Even though equations (6.7) are not identically satisfied, they may be satisfied conditionally and intersector varieties exist. For, any set of values $(U_{0s}, \dots, U_{fm})$ which satisfy equations (6.7) and (6.4) will give an intersector variety.

In this case, we have two varieties V_{n-m+h} and V_{m+h-1} generated by the spaces S_{n-m} and S_{m-1} respectively. On a particular generator of the latter variety we have $(f+1)(m-f-1)$ spaces S_f . Along each of these spaces S_f there is a tangent space $S_{(h+1)(f+1)-1}$ which intersects the corresponding generator S_{n-m} of the variety V_{n-m+h} in a linear space $S_{(f+1)(h+1)-(m+1)+k}$. Each of the varieties V_{h+f} is then, by definition, an intersector variety and the equations of the totality of these intersector varieties are given by the solutions of the differential equations (6.4).

Case (c). $1 < m - k = hf + h + f$. In this case, the determinants of the left members of equations (5.15) have $(h+1)(f+1)$ rows. They may be written in the form

¹Eugenio Bertini, Einführung in die Projective Geometrie Mehrdimensionaler Räume., the German translation by A. Duschek, L. W. Seidel and Son, Vienna, 1924, p. 36.

$$(6.8) \quad \begin{vmatrix} L_{ob}^{(o)} & L_{ib}^{(o)} & L_{fb}^{(h)} \\ L_{ob+1}^{(o)} & L_{ib+1}^{(o)} & \dots L_{f,b+1}^{(h)} \\ \cdot & \cdot & \cdot \\ L_{o,b+m-k}^{(o)} & L_{i,b+m-k}^{(o)} & \dots L_{f,b+m-k}^{(h)} \end{vmatrix} = 0$$

(b = 1, 2, \dots, k),

or else in the form

$$(6.9) \quad \begin{vmatrix} L_{oi}^{(o)} & L_{ii}^{(o)} & \dots L_{fi}^{(h)} \\ L_{os}^{(o)} & L_{is}^{(o)} & \dots L_{fs}^{(h)} \\ \cdot & \cdot & \cdot \\ L_{og}^{(o)} & L_{ig}^{(o)} & \dots L_{fg}^{(h)} \\ L_{o,g+b}^{(o)} & L_{i,g+b}^{(o)} & \dots L_{f,g+b}^{(h)} \end{vmatrix} = 0$$

(b = 1, 2, \dots, k; g = hf + h + f).

By substitution of the values of $L_{pj}^{(q)}$ as given in equations (5.17), equations (6.9) have the form

$$(6.10) \quad \begin{vmatrix} 1 & \dots 1 & \sum_t a_{it} & U_{ot} \dots & \sum_t a_{ith} & U_{ft} \\ U_{os} & \dots U_{fs} & \frac{\partial U_{os}}{\partial u_1} + \sum_t a_{st} & U_{ot} \dots \frac{\partial U_{fs}}{\partial u_h} + \sum_t a_{sth} & U_{ft} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ U_{og} & \dots U_{fg} & \frac{\partial U_{og}}{\partial u_1} + \sum_t a_{gt} & U_{ot} \dots \frac{\partial U_{fg}}{\partial u_h} + \sum_t a_{gth} & U_{ft} \\ U_{o,g+b} \dots U_{f,g+b} & \frac{\partial U_{og+b}}{\partial u_1} + \sum_t a_{g+b,t} & U_{ot} \dots \frac{\partial U_{fg+b}}{\partial u_h} + \sum_t a_{g+b,t,h} & U_{ft} \end{vmatrix} = 0$$

(b = 1, 2, \dots, k; t = 1, 2, \dots, m; g = hf + h + f).

By solving equations (6.10) for any one of the derivatives in the last row, say $\partial U_{f,g+b} / \partial u_h$, we have

$$(6.11) \quad \frac{\partial U_{f,g+b}}{\partial u_h} = F_b \left[u_1, \dots, u_h; U_{p_s}, \dots, U_{p_g}; \right. \\ \left. U_{p,g+1}, \dots, U_{p_m}; \frac{\partial U_{p_s}}{\partial u_q}, \dots, \frac{\partial U_{p_g}}{\partial u_q}; \frac{\partial U_{p_{g+1}}}{\partial u_q}, \dots, \frac{\partial U_{p_m}}{\partial u_q} \right] \\ (p=0,1,\dots,f; q=1,2,\dots,h; b=1,2,\dots,k; g=hf+h+f),$$

where the right member does not contain that particular derivative occurring in the left member. Let

$$(6.12) \quad U_{p_s}, \dots, U_{p_g}; U_{v,g+1}, \dots, U_{v,m} \\ (v=0,1,\dots,f-1; p=0,1,\dots,f; g=hf+h+f),$$

be arbitrary analytic functions of the independent variables $u_1, \dots, u_h$. Then the functions F_b in the right members of equations (6.11) are

$$(6.13) \quad F_b \left[u_1, \dots, u_h, U_{f,g+b}, \frac{\partial U_{f,g+b}}{\partial u_q} \right] \\ (b=1,2,\dots,k; q=1,\dots,h-1; g=hf+h+f),$$

where the variable subscript q assumes all values in its range $1,2,\dots,h$ except that particular value which it has in the left member of equations (6.11). With the functions (6.13) in place of the functions in the right members of equations (6.11), the resulting equations have analytic solutions¹ through an initial point $(u_1^0, u_2^0, \dots, u_h^0; U_{f,g+1}^0, \dots, U_{f,m}^0)$ depending upon the $(m+h)f+h-1$ arbitrary functions (6.12). Thus the following theorem has been proved:

¹Edouard Goursat, A Course in Mathematical Analysis, vol. 2, Part II, of Hedrick's English translation, p. 57.

Theorem 3. In case (c), analytic intersector varieties exist and depend upon at least $(m + h)f + h - 1$ arbitrary functions.

Case (d). $1 < m - k < hf + h + f$. In this case equations (5.15) are unspecialized. By means of equations (5.17), equations (5.15) may be written with U_{pj} as the variables. Solving these equations for one of the derivatives of U_{pj} , say $\partial U_{f,b+m-k} / \partial u_h$, we have, as in case (c)

$$(6.14) \quad \frac{\partial U_{f,b+m-k}}{\partial u_h} = F_b \left[u_1, u_2, \dots, u_h; U_{pj}; \frac{\partial U_{pj}}{\partial u_q} \right]$$

($b=1, \dots, k$; $q=1, \dots, h$; $p=0, 1, \dots, f$; $j=2, 3, \dots, m$),

where the right member does not contain that particular derivative occurring in the left member. Equations (6.14) are k in number and involve $(m - 1)(f + 1)$ variables U_{pj} . Let $(m - 1)(f + 1) - k$ of these variables be arbitrary analytic functions of the independent variables $u_1, \dots, u_h$ and the k remaining variables be dependent functions of them. Then equations (6.14) are in normal form and have analytic solutions.¹ Therefore we have the following

Theorem 4. In case (d), analytic intersector varieties exist and depend upon at least $(m - 1)(f + 1) - k$ arbitrary functions.

The results obtained above may be summarized in the following table:

¹Ibid., p. 57.

Case	$m - k$	$hf + h + f - (m - k)$	Analytic Intersector Varieties	
			Exist	Depend Upon
a	$= 1$	$= 0$	Always	$m - 1$ arbitrary constants
b	$= 1$	> 0	Subject to Integrability Conditions	$(f + 1)(m - f - 1)$ arbitrary constants
c	> 1	$= 0$	Always	At least $(m + h)f + h - 1$ arbitrary functions
d	> 1	> 0	Always	At least $(m - 1)(f + 1) - k$ arbitrary functions

7. Associated families of intersector varieties. Just as the space S_f in the space S_{m-1} generates an intersector variety V_{h+f} of index k on the variety V_{m+h-1} with respect to the variety V_{n-m+h} , so a space S_v in the space S_{n-m} generates an intersector variety V_{h+v} of index k' on the variety V_{n-m+h} with respect to the variety V_{m+h-1} . Proceeding as before to calculate the differential equations which determine the intersector variety V_{h+v} , we merely interchange the roles of S_{m-1} and S_{n-m} , S_f and S_v , k and k' . Let us consider the space S_v as determined by the $v+1$ points $y_0, y_1, \dots, y_v$, where

$$(7.1) \quad y_p = \sum_{j=m+1}^{n+1} \lambda_{pj} x_j \quad (p = 0, 1, \dots, v).$$

As before, the parametric equations giving the points y_p may be written in non-homogeneous coordinates in the form

$$(7.2) \quad y_p = x_{m+1} + \sum_{j=m+2}^{n+1} U_{pj} x_j \quad (p = 0, 1, \dots, v),$$

where

$$(7.3) \quad U_{pj} = \frac{\lambda_{pj}}{\lambda_{p,m+1}}$$

By making the definitions

$$\begin{aligned}
 (7.4) \quad L_{pj}^{(0)} &= \lambda_{pj}, \\
 L_{pj}^{(q)} &= \frac{\partial \lambda_{pj}}{\partial u_q} + \sum_{t=m+1}^{n+1} a_{j tq} \lambda_{pt} \\
 (j=m+1, m+2, \dots, n+1; \quad q=1, 2, \dots, h; \quad p=0, 1, \dots, v)
 \end{aligned}$$

or for the non-homogeneous case

$$\begin{aligned}
 (7.5) \quad L_{pm+1}^{(0)} &= 1, \\
 L_{pj}^{(0)} &= U_{pj}, \\
 L_{pm+1}^{(q)} &= a_{m+1, m+1, q} + \sum_{t=m+2}^{n+1} a_{m+1, t, q} U_{pt}, \\
 L_{pj}^{(q)} &= \frac{\partial U_{pj}}{\partial u_q} + a_{j, m+1, q} + \sum_{t=m+2}^{n+1} a_{j tq} U_{pt} \\
 (j=m+2, \dots, n+1; \quad q=1, \dots, h; \quad p=0, \dots, v),
 \end{aligned}$$

the differential equations of the intersector varieties become

$$\begin{aligned}
 (7.6) \quad & \left| \begin{array}{ccc} L_{B+1, b}^{(A)} & \dots & L_{vb}^{(h)} & L_{Eb}^{(D)} \\ L_{B+1, b+1}^{(A)} & \dots & L_{v, b+1}^{(h)} & L_{E, b+1}^{(D)} \\ \cdot & \cdot & \cdot & \cdot \\ L_{B+1, b+n-m-k'+1}^{(A)} & \dots & L_{v, b+n-m-k'+1}^{(h)} & L_{E, b+n-m-k'+1}^{(D)} \end{array} \right| = 0 \\
 & (b=m+1, m+2, \dots, m+k'; \quad D=0, 1, \dots, A; \quad E=0, 1, \dots, v).
 \end{aligned}$$

This leads to the following

Theorem 5. The space S_v , as defined by equations (7.1)
or equations (7.2), generates an intersector variety V_{h+v} of index
 k' on the variety V_{n-m+h} with respect to the variety V_{m+h-1} if

and only if the functions λ_{pj} or the functions U_{pj} satisfy the differential equations (7.6) where $L_{pj}^{(q)}$ are defined by equations (7.4) or by equations (7.5).

By a proof similar to that of the inequalities (4.11), it is not difficult to prove

$$(7.7) \quad 1 \leq n - m - k' + 1 \leq (v + 1)(h + 1) - 1.$$

Hence, for convenience, we shall divide the discussion of these intersector varieties into four classes, namely

- (a) $1 = n - m - k' + 1 = hv + h + v,$
- (b) $1 = n - m - k' + 1 < hv + h + v,$
- (c) $1 < n - m - k' + 1 = hv + h + v,$
- (d) $1 < n - m - k' + 1 < hv + h + v.$

Case (a). $1 = n - m - k' + 1 = hv + h + v.$ In this case, as previously, it follows that $v = 0$ and we have an $(n - m)$ -parameter family of intersector varieties V_{h+v} . The $(m - 1)$ -parameter family of intersector varieties V_{h+f} on the variety V_{m+h-1} is said to be an associated family¹ of intersector varieties with the $(n - m)$ -parameter family of varieties V_{h+v} on the variety V_{n-m+h} . Should $m - 1 = n - m$, that is, $n = 2m - 1$, the families are called symmetrical associated families² of intersector varieties. The definition of such symmetrical associated families requires that they depend upon the same number of arbitrary constants.

Case (b). $1 = n - m - k' + 1 < hv + h + v.$ Carrying the argument through as before, we find that the intersector varieties exist and depend upon $(v + 1)(n - v - m)$ arbitrary constants, provided that the integrability conditions are satisfied. These

¹W. M. Borgman, loc. cit., p. 30.

²Ibid., p. 31.

integrability conditions are exactly those of equations (6.7) with the subscripts $\underline{i}$ and $\underline{j}$ interchanged on the summation signs.

Case (c). $1 < n - m - k' + 1 = hv + h + v$. Under case (c) the intersector varieties exist and depend upon at least $v(n - m + h + 1) + h - 1$ arbitrary functions.

Case (d). $1 < n - m - k' + 1 < hv + h + v$. In this case, the problem is again unspecialized but the intersector varieties exist and depend upon at least $(n - m)(v + 1) - k'$ arbitrary functions.

We may note that in case (a), $n - m = m - 1$ was sufficient that the associated families of intersector varieties be symmetric. But in case (a), one has $k = k'$, $v = f = 0$. In case (b), the condition $n - m = m - 1$ implies that $k = k'$ but the families are not symmetric unless $v = f$. Hence $v = f$ is a necessary condition for symmetrical associated families. In case (d), the equalities $m - 1 = n - m$, $f = v$ are not sufficient for symmetry but also k must be equal to k' . Since all possibilities have been considered, these conditions are also sufficient and we have

Theorem 6. Necessary and sufficient conditions that two associated families of intersector varieties V_{h+f} and V_{h+v} of index k and k' respectively be symmetrical are that

$$(7.8) \quad v = f, k = k', n = 2m - 1.$$

Symmetrical associated families only exist in linear spaces of odd dimensions.

The last statement of the theorem is due to Borgman¹ and is obvious from the last of equations (7.8).

8. Locus of intersector tangent spaces. Let us choose the $n + 1$ independent points $x_1, \dots, x_{n+1}$ as vertices of a local

¹Ibid.

simplex of reference and choose the unit point so that any point

$$\phi = \phi_1 x_1 + \phi_2 x_2 + \dots + \phi_{n+1} x_{n+1}$$

has coordinates proportional to $(\phi_1, \phi_2, \dots, \phi_{n+1})$. Then if ϕ is any point in the linear tangent space $S_{(f+1)(h+1)-1}$, similar to equation (5.6), we have

$$(8.1) \quad \phi = \sum_{p=0}^f \sum_{j=1}^m \left[\alpha_{po} \lambda_{pj} x_j + \sum_{q=1}^h \alpha_{pq} \left(\frac{\partial \lambda_{pj}}{\partial u_q} x_j + \sum_{i=1}^{n+1} \lambda_{pj} a_{jiq} x_i \right) \right].$$

The coordinates ϕ_1 of the point ϕ are then proportional to the coefficients of the corresponding variable x_1 and may be written

$$(8.2) \quad \rho \phi_r = \sum_{p=0}^f \left[\alpha_{po} \lambda_{pr} + \sum_{q=1}^h \alpha_{pq} \left(\frac{\partial \lambda_{pr}}{\partial u_q} + \sum_{j=1}^m a_{jr q} \lambda_{pj} \right) \right],$$

$$\rho \phi_t = \sum_{p=0}^f \sum_{q=1}^h \sum_{j=1}^m \alpha_{pq} \lambda_{pj} a_{j t q}$$

$$(r = 1, 2, \dots, m; t = m+1, m+2, \dots, n+1),$$

where ρ is a factor of proportionality. Since the variables α , λ are homogeneous, equations (8.2) may be written

$$(8.3) \quad \begin{aligned} \rho \phi_1 &= \sum_{p=0}^f \left[1 + \sum_{q=1}^h \beta_{pq} (a_{11q} + \sum_{j=2}^m a_{j1q} U_{pj}) \right], \\ \rho \phi_r &= \sum_{p=0}^f \left[U_{pr} + \sum_{q=1}^h \beta_{pq} \left(\frac{\partial U_{pr}}{\partial u_q} + a_{1rq} + \sum_{j=2}^m a_{jr q} U_{pj} \right) \right], \\ \rho \phi_t &= \sum_{p=0}^f \sum_{q=1}^h \sum_{j=2}^m \beta_{pq} (a_{j t q} U_{pj} + a_{1 t q}). \end{aligned}$$

$$(r = 2, 3, \dots, m; t = m+1, m+2, \dots, n+1),$$

where the variables α and λ have been replaced as follows:

$$\begin{aligned}
 \alpha_{po} &= 1, \\
 \alpha_{pq} &= \beta_{pq}, \\
 \lambda_{p1} &= 1, \\
 \lambda_{pr} &= U_{pr}.
 \end{aligned}$$

Equations (8.3) are the parametric equations of the locus of the intersector tangent space $S_{(f+1)(h+1)-1}$. For a non-parametric representation of the space, the variables β , U , ρ must be eliminated from equations (8.3). This elimination is not always possible. In cases when the variables can be eliminated the result is $n - (f + 1)(m + h - 1)$ equations in ϕ_i and a_{j1q} which may be represented as

$$(8.4) \quad F_w(\phi_1, \phi_2, \dots, \phi_{n+1}; a_{j1q}) = 0$$

$$(w=(f+1)(m+h-1)+1, \dots, n+1; j=1, \dots, m; i=1, \dots, n+1; q=1, \dots, h).$$

VITA

Virgil Nelson Robinson was born on October 8, 1912, in Richmond, Kentucky. He received his elementary education in Richmond and his high school training from Peabody Demonstration School in Nashville, Tennessee. In 1930, he entered Vanderbilt University, where he received the degrees of A.B. in 1934, and M.A. in 1935. The following two years he taught in the high school system of Atlanta, Georgia. On May 9, 1936, he married Louise Elizabeth Wesley. He entered the University of Chicago in the summer of 1937, where he was in residence for eight quarters. While at the University of Chicago, he studied under Professors Albert, Barnard, Bartky, Bliss, Dickson, Graves, Lane, Logsdon, and Zachariasen.

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